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Numerical investigation of the two– dimensional time–dependent diffusion equation using Radial basis functions

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Abstract

This paper develops a numerical method for solving the partial differential equation in terms of Caputo derivatives with Dirichlet boundary conditions. The approach is based on the two-dimensional Chebyshev wavelet of the second kind with the operational matrix of the collocation method. Furthermore, the convergence and error bound of the proposed method are investigated. For the illustration of the effects of the proposed method, we solve four examples by the presented technique. The obtained results are compared with the results obtained via other numerical methods in which our results are much more accurate than others.

1. Introduction

Much more attention to mathematics, physics, engineering, chemistry and even polymeric materials and social sciences are used fractional calculus with derivatives and integrals of arbitrary orders [1- 4]. It started from the imagination of G.W. Leibniz and L. Euler. Later, Liouville and Abel promulgated the theory of fractional derivatives and integrals in 1823 [5, 6]. Recently, even models of happiness and love have been developed of fractional order and are asserted to give a better presentation than the integer-order [7, 8]. These models often developed of fractional order, and lead to the fractional partial differential equations. Moreover, these equations do not have analytic solutions. We refer the readers to the numerous works for solving these equations of the achieved models that has attracted much attention in recent years.

Chen et al. have solved the fractional diffusion equations by Kansa method which belongs to the Radial basis function (RBF) collocation method [9] and in [10] Fu and Yang have proposed Laplace transformed boundary particle method for these equations. Sun et al. proposed finite difference method for solving variable-order time fractional diffusion equation in [11]. In [12] Heydari et al. have provided two-dimensional legendre wavelets for solving fractional poisson equation with dirichlet boundary conditions. The wave solutions of time-fractional generalized seventh order the Korteweg-de Vries (KdV) equation solved with two-dimensional Legendre wavelet method by Saha et al. in [13]. Also, Saha Ray has based a time-fractional fifth-order Sawada-Kotera (SK) equation on the Chebyshev wavelet expansion in [14].

Nowadays, the ways for solving the partial equation of fractional order are that the unknown can be expanded by a linear combination of the orthogonal basis functions including block puls functions, Legendre, Chebyshev, and Laguerre polynomials also orthogonal wavelets such as Haar, Cas, Legendre, Chebyshev wavelets. In this research, we have applied the second kind of the Chebyshev wavelet for the two class of fractional partial differential equations as

$$\begin{cases} \frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}} + \frac{\partial^{\beta} u(x,t)}{\partial t^{\beta}} = g(x,t), \quad 0 < \alpha, \beta \leq 1 \quad and \quad \alpha, \beta \in \mathbb{R}, \\ u(0,t) = u(x,0) = 0, \end{cases} \tag{1}$$

and

$$\begin{cases} \frac{\partial^{\alpha} v(x,y)}{\partial x^{\alpha}} + \frac{\partial^{\beta} v(x,y)}{\partial y^{\beta}} = f(x,y), & 1 < \alpha, \beta \leq 2, \quad and \quad \alpha, \beta \in \mathbb{R}, \\ u(x,0) = f_1(x), u(0,y) = g_1(y), & boundaryconditions, \\ u(x,1) = f_2(x), u(1,y) = g_2(y), & (x,y) \in [0,1] \times [0,1]. \end{cases} \tag{2}$$

where $\frac{\partial^{\alpha} u(x,y)}{\partial x^{\alpha}}$ and $\frac{\partial^{\beta} u(x,y)}{\partial x^{\beta}}$ are fractional derivatives of Caputo sense, $g(x,t)$ and $f(x,y)$ are the known continuous functions, $u(x,t)$ and $v(x,y)$ are the unknown functions.

Several methods are used to solve the fractional partial equation such as the Laplace transform [18], the block pulse operational matrix [19], the Hermite wavelets [20] and an implicit fully discrete local discontinuous Galerkin method [21]. We focuse of using the second kind of Chebyshev wavelet on collocation spectral method and we investigate convergence analysis too. The main advantage of the Chebyshev wavelet is the reduction of the collocation method to a system of linear or nonlinear algebraic, resulting in speeding up the computation of the performance. Other important advantages are the use of wavelet that wavelets possess the ability to detect singularities, the exact representation of polynomials to a certain degree and compact support.

This paper is partitioned into the following sections: the second section introduces preparations and notations of the two-dimensional fractional derivatives. The two-dimensional Chebyshev wavelets of the second kind and their properties are explained in the next section. In the fourth section, we summarized the collocation method for solving the fractional partial differential equation involving Caputo derivatives. The convergence analysis and the upper error bound are presented in the fifth section. The accuracy of the approach is shown by solving four numerical examples in the residue of this paper.

2 Preliminaries and Notations of Fractional Derivatives

To complete the explanations, we recollect the concept of a fractional derivative, which can be found in [22, 5]. Three definitions of fractional derivatives are widely used. The GrÃ¼nwald-Letnikov derivative, the Riemann-Liouville derivative and the Caputo derivative are in general non equivalence. We use the Caputo derivative because it only requires initial conditions given in terms of integer-order derivatives. We recommend some definitions to the facts to two-dimensions fractional calculus which will be used further in this work.

The two-dimensions integral operator of the Rieman-liouville $\mathcal{J}^{\beta}$ of order $\beta \in \mathbb{R}^{+}$ is given by [22, 23]

$$(\mathcal{J}^{\beta} u)(x,t) = \begin{cases} \frac{1}{\Gamma(\beta)} \int_0^x (x-\tau)^{\beta-1} u(\tau,t) d\tau, & \beta \in \mathbb{R}^{+}, \\ u(x,t), & \beta = 0, \end{cases}$$

some properties of the operator $\mathcal{J}^{\beta}$ are followed as:

$$\mathcal{J}^{\beta}\mathcal{J}^{\alpha}u(x,t)=\mathcal{J}^{\alpha}\mathcal{J}^{\beta}u(x,t)=\mathcal{J}^{\beta+\alpha}u(x,t),$$

$$\mathcal{J}^{\beta}(x-a)^{\mathfrak{r}}=\frac{\Gamma(\mathfrak{r}+1)}{\Gamma(\beta+\mathfrak{r}+1)}(x-a)^{\beta+\mathfrak{r}},$$

$$\mathcal{J}^{\beta}(\gamma f(x,t)+\eta g(x,t))=\gamma\mathcal{J}^{\beta}f(x,t)+\eta\mathcal{J}^{\beta}g(x,t),\quad 0.1cm\gamma,\delta\in\mathbb{C},$$

The Rieman-liouville derivative has disutility when trying to model real occurrence. Therefore, we may introduce a modified Rieman-liouville derivative nominated by Caputo. We proposed Caputo's derivative of fractional order $\beta\in\mathbb{R}^{+}$ as [24]

$$ {}_{0}^{C}\mathbb{D}_{x}^{\beta}u(x,t)=\begin{cases}\frac{1}{\Gamma(n-\beta)}\int_{0}^{x}(x-\tau)^{n-\beta-1}\frac{\partial^{n}u(\tau,t)}{\partial\tau^{n}}d\tau, & n-1<\beta<n\in\mathbb{N},\\ \frac{\partial^{n}u(x,t)}{\partial x^{n}}, & \beta=n,\end{cases} \tag{3}$$

The following are some properties of the Caputo fractional derivative ${}_{0}^{C}\mathbb{D}_{t}^{\beta}$

1. $\mathcal{J}^{\beta}\mathbb{D}^{\beta}u(x,t)=u(x,t)-\sum_{k=0}^{n-1}\frac{\partial^{k}u(0^{+},t)}{\partial x^{k}}\frac{x^{k}}{k!},\quad n-1<\beta\leq n\in\mathbb{N},$
2. $\lim_{\beta\to n}\mathbb{D}^{\beta}u(x,t)=\frac{\partial^{n}u(x,t)}{\partial x^{n}},\ \lim_{\beta\to n-1}\mathbb{D}^{\beta}g(t)=\frac{\partial^{n-1}u(x,t)}{\partial x^{n-1}}-\frac{\partial^{n-1}u(0,t)}{\partial x^{n-1}},$
3. $\mathbb{D}^{\beta}x^{\alpha}=\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-\beta)}x^{\alpha-\beta},\quad 0<\beta<\alpha+1,\beta>-1,$
4. $\mathbb{D}^{\beta}(\gamma f(x,t)+\eta g(x,t))=\gamma\mathbb{D}^{\beta}f(x,t)+\eta\mathbb{D}^{\beta}g(x,t),$
5. $\mathbb{D}^{\beta}\mathbb{D}^{n}g(x,t)=\mathbb{D}^{\beta+n}g(x,t)\neq\mathbb{D}^{n}\mathbb{D}^{\beta}g(x,t),$
6. $\mathbb{D}^{\beta}(c)=0, cisconstant.$

3 The second kind of two-dimensional Chebyshev wavelets

The mother wavelet $\psi(x)$ comprises a classification of functions produced from dilation and translation of single function [25].

$$\boldsymbol{\psi}_{\mathfrak{a},\mathfrak{b}}(x)=|\mathfrak{a}|^{-\frac{1}{2}}\boldsymbol{\psi}\left(\frac{x-\mathfrak{b}}{\mathfrak{a}}\right),\quad \mathfrak{a},\mathfrak{b}\in\mathbb{R},\mathfrak{a}\neq 0, \tag{4}$$

where the parameters $\mathfrak{a}$ and $\mathfrak{b}$ are defined as $\mathfrak{a}={\mathfrak{a}_0}^{-\mathfrak{z}}$, $\mathfrak{b}=\mathfrak{n}\mathfrak{b}_0{\mathfrak{a}_0}^{-\mathfrak{z}}, \mathfrak{a}_0>1, \mathfrak{b}_0>0$ and denominate dilation and translation parameter respectively. So, we structure a wavelet basis for $\mathbb{L}_2(\mathbb{R})$ which has the following family of discrete wavelets

$$\boldsymbol{\psi}_{\mathfrak{n},\mathfrak{m}}(x)=|\mathfrak{a}_0|^{\frac{\mathfrak{n}}{2}}\boldsymbol{\psi}({\mathfrak{a}_0}^{\mathfrak{n}}x-\mathfrak{m}\mathfrak{b}_0),\quad \mathfrak{n},\mathfrak{m}\in\mathbb{Z}, \tag{5}$$

If $\mathfrak{a}_0=2$ and $\mathfrak{b}_0=1$, then $\boldsymbol{\psi}_{\mathfrak{n},\mathfrak{m}}(x)$ creates an orthonormal basis. The Chebyshev polynomials of the second kind $\boldsymbol{\psi}_{\mathfrak{n},\mathfrak{m}}(x)=\boldsymbol{\psi}(\mathfrak{k},\tilde{\mathfrak{n}},\mathfrak{m},x)$ have four bulding, $\tilde{\mathfrak{n}}=2\mathfrak{n}-1$, $\mathfrak{n}=1,2,\ldots,2^{\mathfrak{k}-1}$, $\mathfrak{k}$ is any positive integer, $\mathfrak{m}=0,1,2,\ldots,\mathfrak{M}-1$, is the order for the Chebyshev polynomial. They are described on the interval $[0,1)$ as

$$\boldsymbol{\psi}_{\mathfrak{n},\mathfrak{m}}(x)=\begin{cases}\sqrt{\frac{2^{\mathfrak{k}+1}}{\pi}}\mathfrak{U}_{\mathfrak{m}}(2^{\mathfrak{k}}x-\tilde{\mathfrak{n}}), & \frac{\mathfrak{n}-1}{2^{\mathfrak{k}-1}}\leq x<\frac{\mathfrak{n}}{2^{\mathfrak{k}-1}},\\ 0, & otherwise,\end{cases} \tag{6}$$

Here $\mathfrak{U}_{\mathfrak{m}}(x)$ are the second kind of Chebyshev polynomials of degree $\mathfrak{m}$ which are orthonormal with respect to the weight function $\mathfrak{W}_{\mathfrak{n},\mathfrak{k}}(x)=\mathfrak{W}(2^{\mathfrak{k}}x-\tilde{\mathfrak{n}})$ on the interval $[-1,1]$ where $\mathfrak{W}(x)=\sqrt{1-x^2}$. We can introduce the set of two-dimensional Chebyshev wavelets which forms an orthogonal basis of $\mathbb{L}_2(\mathbb{R}\times\mathbb{R})$. The two-dimensional Chebyshev wavelets of the second kind are emblazoned as

$$\boldsymbol{\psi}_{\mathfrak{n}_1,\mathfrak{m}_1,\mathfrak{n}_2,\mathfrak{m}_2}(x,t)=\begin{cases}A\mathfrak{U}_{\mathfrak{m}_1}(2^{\mathfrak{k}_1}x-\tilde{\mathfrak{n}}_1)\mathfrak{U}_{\mathfrak{m}_2}(2^{\mathfrak{k}_2}x-\tilde{\mathfrak{n}}_2), & \frac{\mathfrak{n}_1-1}{2^{\mathfrak{k}_1-1}}\leq x<\frac{\mathfrak{n}_1}{2^{\mathfrak{k}_1-1}},\\ & \frac{\mathfrak{n}_2-1}{2^{\mathfrak{k}_2-1}}\leq t<\frac{\mathfrak{n}_2}{2^{\mathfrak{k}_2-1}},\\ 0, & otherwise,\end{cases} \tag{7}$$

where $A=\frac{2}{\pi}\sqrt{2^{\mathfrak{k}_1+\mathfrak{k}_2}}$, $\mathfrak{n}_1,\mathfrak{n}_2,\mathfrak{k}_1$ and $\mathfrak{k}_2$ are defined similarly to $\mathfrak{n}$ and $\mathfrak{k}$ respectively. The order for two-dimensions Chebyshev wavelets of the second kind are $\mathfrak{m}_1$ and $\mathfrak{m}_2$ and $\boldsymbol{\psi}_{\mathfrak{n}_1,\mathfrak{m}_1,\mathfrak{n}_2,\mathfrak{m}_2}(x,t)$ construct a basis for $\mathbb{L}_2([0,1]\times[0,1])$. The second kind of two-dimensions Chebyshev wavelets are orthogonal with respect to the weight function $\omega(x,t)=\mathfrak{W}_{\mathfrak{n}_1,\mathfrak{k}_1}(x)\times\mathfrak{W}_{\mathfrak{n}_2,\mathfrak{k}_2}(t)$ on the interval $[0,1]\times[0,1]$.

$$
\begin{aligned}
&\left\langle\boldsymbol{\psi}_{\mathfrak{n}_1,\mathfrak{m}_1,\mathfrak{n}_2,\mathfrak{m}_2}(x,t),\boldsymbol{\psi}_{\mathfrak{i}_1,\mathfrak{j}_1,\mathfrak{i}_2,\mathfrak{j}_2}(x,t)\right\rangle_{\mathbb{L}_2([0,1]\times[0,1]),\omega(x,t)}\\
&=\int_0^1\int_0^1\omega(x,t)\boldsymbol{\psi}_{\mathfrak{n}_1,\mathfrak{m}_1,\mathfrak{n}_2,\mathfrak{m}_2}(x,t)\boldsymbol{\psi}_{\mathfrak{i}_1,\mathfrak{j}_1,\mathfrak{i}_2,\mathfrak{j}_2}(x,t)dxdt=\delta_{\mathfrak{n}_1,\mathfrak{i}_1}\delta_{\mathfrak{m}_1,\mathfrak{j}_1}\delta_{\mathfrak{n}_2,\mathfrak{i}_2}\delta_{\mathfrak{m}_2,\mathfrak{j}_2},
\end{aligned}
$$

where $\delta_{\mathfrak{n},\mathfrak{i}}$ and $\delta_{\mathfrak{m},\mathfrak{j}}$ are Kronecker delta . A function $u(x,t)\in\mathbb{L}_2([0,1]\times[0,1])$ may be expanded by the second kind of two-dimensions Chebyshev wavelets as

$$
u(x,t)=\sum_{\mathfrak{n}_1=1}^{\infty}\sum_{\mathfrak{m}_1=0}^{\infty}\sum_{\mathfrak{n}_2=1}^{\infty}\sum_{\mathfrak{m}_2=0}^{\infty}\mathfrak{c}_{\mathfrak{n}_1,\mathfrak{m}_1,\mathfrak{n}_2,\mathfrak{m}_2}\boldsymbol{\psi}_{\mathfrak{n}_1,\mathfrak{m}_1,\mathfrak{n}_2,\mathfrak{m}_2}(x,t)=u_{\mathfrak{n}_1\mathfrak{m}_1\mathfrak{n}_2\mathfrak{m}_2}(x,t),\qquad(8)
$$

where $\mathfrak{c}_{\mathfrak{n}_1,\mathfrak{m}_1,\mathfrak{n}_2,\mathfrak{m}_2}=\left\langle u(x,t),\boldsymbol{\psi}_{\mathfrak{n}_1,\mathfrak{m}_1,\mathfrak{n}_2,\mathfrak{m}_2}(x,t)\right\rangle_{\mathbb{L}_2([0,1]\times[0,1]),\omega(x,t)}$. If the infinite series in Eq. (8) is truncated, then it can be written as

$$
u(x,t)\approx\sum_{\mathfrak{n}_1=1}^{2^{\mathfrak{k}_2-1}}\sum_{\mathfrak{m}_1=0}^{\mathfrak{M}_1-1}\sum_{\mathfrak{n}_2=1}^{2^{\mathfrak{k}_1-1}}\sum_{\mathfrak{m}_2=0}^{\mathfrak{M}_2-1}\mathfrak{c}_{\mathfrak{n}_1,\mathfrak{m}_1,\mathfrak{n}_2,\mathfrak{m}_2}\boldsymbol{\psi}_{\mathfrak{n}_1,\mathfrak{m}_1,\mathfrak{n}_2,\mathfrak{m}_2}(x,t)=\tilde{u}(x,t)=\boldsymbol{\Psi}^T(x)\mathfrak{C}\boldsymbol{\Psi}(t),\quad(9)
$$

where dimensions $\boldsymbol{\Psi}(x),\boldsymbol{\Psi}(t)$ and $\mathfrak{C}$ are $2^{\mathfrak{k}_1-1}\times(\mathfrak{M}_1-1),2^{\mathfrak{k}_2-1}\times(\mathfrak{M}_2-1)$ and $2^{\mathfrak{k}_1-1}\mathfrak{M}_1\times2^{\mathfrak{k}_2-1}\mathfrak{M}_2$ matrics respectively given by

$$
\begin{aligned}
\boldsymbol{\Psi}(x)=&\quad[\psi_{10}(x),\ldots,\psi_{1\mathfrak{M}_1-1},\psi(x)_{20},\ldots,\psi(x)_{2\mathfrak{M}_1-1},\ldots,\psi(x)_{2^{\mathfrak{k}_1-1}0},\ldots,\psi(x)_{2^{\mathfrak{k}_1-1}\mathfrak{M}_1-1}]^T,\\
\boldsymbol{\Psi}(t)=&\quad[\psi_{10}(t),\ldots,\psi_{1\mathfrak{M}_2-1},\psi(t)_{20},\ldots,\psi(t)_{2\mathfrak{M}_2-1},\ldots,\psi(t)_{2^{\mathfrak{k}_2-1}0},\ldots,\psi(t)_{2^{\mathfrak{k}_2-1}\mathfrak{M}_2-1}]^T.
\end{aligned}\qquad(10)
$$

Two common methods for determining the coefficients $\mathfrak{c}_{\mathfrak{n}_1,\mathfrak{m}_1,\mathfrak{n}_2,\mathfrak{m}_2}$ are through the use of the inner product as in Galerkin's method and through the use of collocation which is related to interpolation. In this article, we use the collocation method. This collection points corresponds $x_p=\frac{p-1}{2^{\mathfrak{k}_1-1}\mathfrak{M}_1-1}$ and $t_q=\frac{q-1}{2^{\mathfrak{k}_2-1}\mathfrak{M}_2-1}$ for $p=1,2,\ldots,\mathfrak{N}_1=2^{\mathfrak{k}_1-1}\mathfrak{M}_1$ and $q=1,2,\ldots,\mathfrak{N}_2=2^{\mathfrak{k}_2-1}\mathfrak{M}_2$.

4 Explanation of The Chebyshev Wavelet Collocation Scheme

In this section, we meant to apply the second kind of Chebyshev wavelet method for the numerical solution of fractional partial differential equations (1) and (2). To solve these problems we approximate $u(x,t)=\boldsymbol{\Psi}^T(x)\mathfrak{C}\boldsymbol{\Psi}(t)$, where $\mathfrak{C}=[c_{i,j}]_{\widetilde{\mathfrak{m}}\times\widetilde{\mathfrak{m}}}$ is an unknown matrix and $\boldsymbol{\Psi}(.)$ is the vector that is defined in Eq. (10). By fractional derivative definition of order α and β of Eq. (3) with respect to x and t respectively when $n-1<\alpha,\beta<n\in\mathbb{N}$, we obtain

$$
\begin{aligned}
\frac{\partial^\alpha u(x,t)}{\partial x^\alpha}&\approx\frac{1}{\Gamma(n-\alpha)}\int_0^x(x-\tau)^{n-\alpha-1}\frac{\partial^n(\boldsymbol{\Psi}^T(\tau)\mathfrak{C}\boldsymbol{\Psi}(t))}{\partial\tau^n}d\tau=\tilde{\mathfrak{J}}_x^\alpha u(x,t),\\
\frac{\partial^\beta u(x,t)}{\partial t^\alpha}&\approx\frac{1}{\Gamma(n-\beta)}\int_0^t(t-\tau)^{n-\beta-1}\frac{\partial^n(\boldsymbol{\Psi}^T(x)\mathfrak{C}\boldsymbol{\Psi}(\tau))}{\partial\tau^n}d\tau=\tilde{\mathfrak{J}}_t^\beta u(x,t).
\end{aligned}\qquad(11)
$$

We can also apply the approximated method for the boundary and initial conditions in (1) as

$$
\begin{cases}u(0,t)\approx\boldsymbol{\Psi}^T(0)\mathfrak{C}\boldsymbol{\Psi}(t),\\u(x,0)\approx\boldsymbol{\Psi}^T(x)\mathfrak{C}\boldsymbol{\Psi}(0),\end{cases}\qquad(12)
$$

respectively, and the boundary conditions in (2) as

$$\begin{cases} u(x,0) = f_1(x) \approx \boldsymbol{\Psi}^T(x)\mathfrak{C}\boldsymbol{\Psi}(0), \quad u(0,y) = g_1(y) \approx \boldsymbol{\Psi}^T(0)\mathfrak{C}\boldsymbol{\Psi}(y), \\ u(x,1) = f_2(x) \approx \boldsymbol{\Psi}^T(x)\mathfrak{C}\boldsymbol{\Psi}(1), \quad u(1,y) = g_2(y) \approx \boldsymbol{\Psi}^T(1)\mathfrak{C}\boldsymbol{\Psi}(y). \\ \end{cases} \tag{13}$$

Applying Eqs. (11) and (12) in (1) and substituting the collocation nodes x_p and t_q for $p = 1,2,\dots,\mathfrak{N}_1$ and $q = 1,2,\dots,\mathfrak{N}_2$, we get $\mathfrak{N}_1 \times \mathfrak{N}_2$ equations in $\mathfrak{N}_1 \times \mathfrak{N}_2$ unknown coefficients. The linear system (13) can be written as

$$\begin{cases} \tilde{\mathfrak{J}}_x^{\alpha} u(x_p,t_q) + \tilde{\mathfrak{J}}_t^{\beta} u(x_p,t_q) = g(x_p,t_q), \quad 0 < \alpha,\beta \le 1 \quad and \quad \alpha,\beta \in \mathbb{R}, \\ \boldsymbol{\Psi}^T(0)\mathfrak{C}\boldsymbol{\Psi}(t_q) = \boldsymbol{\Psi}^T(x_p)\mathfrak{C}\boldsymbol{\Psi}(0) = 0, \quad initialconditions. \end{cases} \tag{14}$$

Also, applying Eqs. (11) and (13) in (2) and substituting the collocation nodes $x_p = \frac{p-1}{\mathfrak{N}_1-1}$ and $y_q = \frac{q-1}{\mathfrak{N}_2-1}$ for $p = 1,2,\dots,\mathfrak{N}_1$ and $q = 1,2,\dots,\mathfrak{N}_2$, we get the linear system $\mathcal{A}\mathcal{X} = \mathcal{B}$, where

$$\mathcal{A} = \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \\ \Lambda_3 \\ \end{bmatrix}_{\mathfrak{N}_1\mathfrak{N}_2 \times \mathfrak{N}_1\mathfrak{N}_2}, \quad \mathcal{B} = \begin{bmatrix} \mathcal{B}_1 \\ \mathcal{B}_2 \\ \mathcal{B}_3 \\ \end{bmatrix}_{\mathfrak{N}_1\mathfrak{N}_2 \times 1},$$

$$\Lambda_1 = [\lambda_{1,j}^1]_{1\times\mathfrak{N}_1\mathfrak{N}_2}, \lambda_{1,j}^1 = \Psi_{pq}(0)\Psi_{qp}(y_j), \quad \Lambda_3 = [\lambda_{1,j}^3]_{1\times\mathfrak{N}_1\mathfrak{N}_2}, \lambda_{1,j}^3 = \Psi_{pq}(1)\Psi_{qp}(y_j),$$

$$\Lambda_2 = [\Upsilon_1 \quad \Upsilon_2 \quad \Upsilon_3]_{(\mathfrak{N}_1\mathfrak{N}_2-2)\times(\mathfrak{N}_1\mathfrak{N}_2-2)},$$

$$\Upsilon_1 = [v_{i,1}^1]_{(\mathfrak{N}_1\mathfrak{N}_2-2)\times 1}, \quad v_{i,1}^1 = \Psi_{pq}(x_i)\Psi_{qp}(0), \quad \Upsilon_3 = [v_{i,1}^3]_{(\mathfrak{N}_1\mathfrak{N}_2-2)\times 1}, \quad v_{i,1}^3 = \Psi_{pq}(x_i)\Psi_{qp}(1),$$

$$\Upsilon_2 = [v_{i,j}^2]_{(\mathfrak{N}_1\mathfrak{N}_2-2)\times(\mathfrak{N}_1\mathfrak{N}_2-2)}, \quad v_{i,j}^2 = \tilde{\mathfrak{J}}_x^{\alpha} v(x_i,y_j) + \tilde{\mathfrak{J}}_y^{\beta} v(x_i,y_j)$$

$$\mathcal{B}_1 = [b_{1,1}^1]_{1\times 1}, \quad b_{1,1}^1 = g_1(y_j), \quad \mathcal{B}_3 = [b_{1,1}^3]_{1\times 1}, \quad b_{1,1}^3 = g_2(y_j),$$

$$\mathcal{B}_2^1 = [b_{i,j}^2]_{(\mathfrak{N}_1\mathfrak{N}_2-2)\times 1}, \quad b_{i,j}^2 = \Psi_{pq}(x_i)\Psi_{qp}(0) + f(x_i,y_j) + \Psi_{pq}(x_i)\Psi_{qp}(1), \quad i,j = 2,3,\dots,\mathfrak{N}_1\mathfrak{N}2-1.$$

5 The error approximation

In this section, the convergence analysis of the two-dimensional Chebyshev wavelet of the second kind is explored for the fractional partial differential equations. Firstly, we express the following theorem.

Theorem 1 *(See [25]). Suppose* $f(x)$ *defined on* $[0,1)$ *with bounded second derivative* $|f''(x)| < C$. *Then It can be expanded as an infinite sum of chebyshev wavelets and the series converges uniformaly to* $f(x)$, *that is*

$$f(x) = \sum_{\mathfrak{n}=1}^{\infty}\sum_{\mathfrak{m}=0}^{\infty} \mathfrak{c}_{\mathfrak{n}\mathfrak{m}} \boldsymbol{\psi}_{\mathfrak{n},\mathfrak{m}}(x), \tag{15}$$

where $\mathfrak{c}_{\mathfrak{n}\mathfrak{m}} = \langle f(x), \boldsymbol{\psi}_{\mathfrak{n},\mathfrak{m}}(x)\rangle_{\mathfrak{W}_{\mathfrak{n},\mathfrak{k}}(x)}$ and C is constant.

Theorem 2 *Suppose a continuous function* $u(x,y) \in \mathbb{L}_2(\mathbb{R}\times\mathbb{R})$ *defined on* $[0,1)\times[0,1)$ *with bounded second mixed fourth partial derivative* $|\frac{\partial^i u(x,y)}{\partial x^j \partial y^j}| < \mathcal{M}_{j,j}$, *then the two-dimensional Chebyashev wavelet of the second kind approximation of* $u(x,y)$ *converges to it and also*

$$\left|\mathfrak{c}_{\mathfrak{n}_1,\mathfrak{m}_1,\mathfrak{n}_2,\mathfrak{m}_2}\right| \le \frac{\widetilde{\mathcal{M}}}{\mathfrak{m}_1{}^3\mathfrak{m}_2{}^3(\mathfrak{n}_1\mathfrak{n}_2)^{\frac{3}{2}}(\mathfrak{n}_1\mathfrak{n}_2)^{\frac{3}{2}}}, \tag{16}$$

where $\mathcal{M}_{j,j}$ is constant and $\widetilde{\mathcal{M}} = Max\{\mathcal{M}_{j,j}\}, i = 1,2,3,4, j = 1,2$.

Proof. The two-dimensional Chebyshev wavelet coefficients $\mathfrak{c}_{\mathfrak{n}_1,\mathfrak{m}_1,\mathfrak{n}_2,\mathfrak{m}_2} = \mathfrak{c}_{\mathbf{n},\mathbf{m}}$ of continuous function $u(x,y)$ relative to the weight function $\omega(x,y)$ are obtained

$$\begin{aligned}\mathfrak{c}_{\mathbf{n},\mathbf{m}} &= \int_0^1\int_0^1 \omega(x,y)\boldsymbol{\psi}_{\mathfrak{n}_1,\mathfrak{m}_1,\mathfrak{n}_2,\mathfrak{m}_2}(x,y)u(x,y)dxdy,\\ &= \int_0^1\int_{\frac{\mathfrak{n}_1-1}{2^{\mathfrak{k}_1-1}}}^{\frac{\mathfrak{n}_1}{2^{\mathfrak{k}_1-1}}} A\omega\big(2^{\mathfrak{k}_1}x-\tilde{\mathfrak{n}}_1,y\big)\mathfrak{U}_{\mathfrak{m}_1}\big(2^{\mathfrak{k}_1}x-\tilde{\mathfrak{n}}_1\big)\mathfrak{U}_{\mathfrak{m}_2}\big(2^{\mathfrak{k}_2}y-\tilde{\mathfrak{n}}_2\big)u(x,y)dxdy, \qquad (17)\end{aligned}$$

Now by the definition of the weight function $\omega(x,y)$, the basis function $\boldsymbol{\psi}_{\mathfrak{n}_1,\mathfrak{m}_1,\mathfrak{n}_2,\mathfrak{m}_2}(x,y)$ change of variables $2^{\mathfrak{k}_1}x-\tilde{\mathfrak{n}}_1=\mathrm{t}$, we get

$$\begin{aligned}\mathfrak{c}_{\mathbf{n},\mathbf{m}} &= \int_0^1\int_{-1}^1 \frac{A}{2^{\mathfrak{k}_1}}\omega(\frac{\mathrm{t}+\tilde{\mathfrak{n}}_1}{2^{\mathfrak{k}_1}},y)\mathfrak{U}_{\mathfrak{m}_1}(\mathrm{t})\mathfrak{U}_{\mathfrak{m}_2}(2^{\mathfrak{k}_2}y-\tilde{\mathfrak{n}}_2)u(\frac{\mathrm{t}+\tilde{\mathfrak{n}}_1}{2^{\mathfrak{k}_1}},y)d\mathrm{t}dy,\\ &= \frac{A}{2^{\mathfrak{k}_1}}\int_0^1 \mathfrak{W}_{\mathfrak{n}_2,\mathfrak{k}_2}(y)\mathfrak{U}_{\mathfrak{m}_2}(2^{\mathfrak{k}_2}y-\tilde{\mathfrak{n}}_2)\int_{-1}^1 \mathfrak{W}_{\mathfrak{n}_1,\mathfrak{k}_1}(\frac{\mathrm{t}+\tilde{\mathfrak{n}}_1}{2^{\mathfrak{k}_1}})\mathfrak{U}_{\mathfrak{m}_1}(\mathrm{t})u(\frac{\mathrm{t}+\tilde{\mathfrak{n}}_1}{2^{\mathfrak{k}_1}},y)d\mathrm{t}dy,\end{aligned} \qquad (18)$$

Now, we can substitute $\mathfrak{U}_{\mathfrak{m}_1}(\mathrm{t})d\mathrm{t}=\frac{1}{\mathfrak{m}_1+1}d\mathfrak{T}_{\mathfrak{m}_1+1}(\mathrm{t})$ where $\mathfrak{T}_{\mathfrak{m}_1}(\mathrm{t})$ is the Chebyshev polynomials of the first kind and $\mathfrak{W}_{\mathfrak{n}_1,\mathfrak{k}_1}(\frac{\mathrm{t}+\tilde{\mathfrak{n}}_1}{2^{\mathfrak{k}_1}})=\mathfrak{W}(\mathrm{t})=\sqrt{1-\mathrm{t}^2}$ where $\mathfrak{W}(\mathrm{t})$ is weight function of the second kind of Chebyahev wavelet, then

$$\begin{aligned}\mathfrak{c}_{\mathbf{n},\mathbf{m}} &= \frac{A}{(\mathfrak{m}_1+1)\times 2^{\mathfrak{k}_1}}(\mathfrak{W}_{\mathfrak{n}_2,\mathfrak{k}_2}(y)\mathfrak{U}_{\mathfrak{m}_2}(2^{\mathfrak{k}_2}y-\tilde{\mathfrak{n}}_2)(\mathfrak{W}(\mathrm{t})u(\frac{\mathrm{t}+\tilde{\mathfrak{n}}_1}{2^{\mathfrak{k}_1}},y)\mathfrak{T}_{\mathfrak{m}_1+1}(\mathrm{t})|_{-1}^{1})\\ &\quad -\frac{A}{(\mathfrak{m}_1+1)\times 2^{\mathfrak{k}_1}}\int_0^1 \mathfrak{W}_{\mathfrak{n}_2,\mathfrak{k}_2}(y)\mathfrak{U}_{\mathfrak{m}_2}(2^{\mathfrak{k}_2}y-\tilde{\mathfrak{n}}_2)\int_{-1}^1(\mathfrak{W}'(\mathrm{t})u\left(\frac{\mathrm{t}+\tilde{\mathfrak{n}}_1}{2^{\mathfrak{k}_1}},y\right) \qquad (19)\\ &\quad +\frac{1}{2^{\mathfrak{k}_1}}\mathfrak{W}(\mathrm{t})\frac{\partial u(\frac{\mathrm{t}+\tilde{\mathfrak{n}}_1}{2^{\mathfrak{k}_1}},y)}{\partial\mathrm{t}}\mathfrak{T}_{\mathfrak{m}_1+1}(\mathrm{t}))d\mathrm{t}dy.\end{aligned}$$

The first sentence of the above equation is zero because $\mathfrak{W}(1)=\mathfrak{W}(-1)=0$. Now, we apply $\mathfrak{T}_{\mathfrak{m}_1+1}(\mathrm{t})d\mathrm{t}=\frac{1}{\mathfrak{m}_1+1}((\mathrm{t}^2-1)d\mathfrak{U}_{\mathfrak{m}_1}(\mathrm{t})+\mathrm{t}\mathfrak{U}_{\mathfrak{m}_1}(\mathrm{t})d\mathrm{t})$ and after simplifying, we have

$$\begin{aligned}\mathfrak{c}_{\mathbf{n},\mathbf{m}} &= \frac{-A}{2^{\mathfrak{k}_1}(\mathfrak{m}_1+1)^2}\int_0^1 \mathfrak{W}_{\mathfrak{n}_2,\mathfrak{k}_2}(y)\mathfrak{U}_{\mathfrak{m}_2}(2^{\mathfrak{k}_2}y-\tilde{\mathfrak{n}}_2)(\int_{-1}^1 \mathrm{t}\mathfrak{W}(\mathrm{t})u(\frac{\mathrm{t}+\tilde{\mathfrak{n}}_1}{2^{\mathfrak{k}_1}},y)d\mathfrak{U}_{\mathfrak{m}_1}(\mathrm{t})\\ &\quad +\int_{-1}^1 \mathrm{t}\mathfrak{W}'(\mathrm{t})u\left(\frac{\mathrm{t}+\tilde{\mathfrak{n}}_1}{2^{\mathfrak{k}_1}},y\right)\mathfrak{U}_{\mathfrak{m}_1}(\mathrm{t})d\mathrm{t}-\frac{1}{2^{\mathfrak{k}_1}}\int_{-1}^1 \mathfrak{W}^3(\mathrm{t})\frac{\partial u\left(\frac{\mathrm{t}+\tilde{\mathfrak{n}}_1}{2^{\mathfrak{k}_1}},y\right)}{\partial\mathrm{t}}d\mathfrak{U}_{\mathfrak{m}_1}(\mathrm{t}) \qquad (20)\\ &\quad +\frac{1}{2^{\mathfrak{k}_1}}\int_{-1}^1 \mathrm{t}\mathfrak{W}(\mathrm{t})\frac{\partial u(\frac{\mathrm{t}+\tilde{\mathfrak{n}}_1}{2^{\mathfrak{k}_1}},y)}{\partial\mathrm{t}}\mathfrak{U}_{\mathfrak{m}_1}(\mathrm{t})d\mathrm{t})dy.\end{aligned}$$

So simplifying the above equation and using equality (18), we get

$$\begin{aligned}\mathfrak{c}_{\mathfrak{n}\mathfrak{m}} &= \frac{A\times 2^{-\mathfrak{k}_1}}{1+(\mathfrak{m}_1+1)^2}\int_0^1 \mathfrak{W}_{\mathfrak{n}_2,\mathfrak{k}_2}(y)\mathfrak{U}_{\mathfrak{m}_2}(2^{\mathfrak{k}_2}y-\tilde{\mathfrak{n}}_2)[\frac{3}{2^{\mathfrak{k}_1}}\int_{-1}^1 \mathrm{t}\mathfrak{W}(\mathrm{t})\frac{\partial u(\frac{\mathrm{t}+\tilde{\mathfrak{n}}_1}{2^{\mathfrak{k}_1}},y)}{\partial\mathrm{t}}\mathfrak{U}_{\mathfrak{m}_1}(\mathrm{t})d\mathrm{t}\\ &\quad -\frac{1}{2^{2\mathfrak{k}_1}}\int_{-1}^1 \mathfrak{W}^3(\mathrm{t})\frac{\partial^2 u(\frac{\mathrm{t}+\tilde{\mathfrak{n}}_1}{2^{\mathfrak{k}_1}},y)}{\partial\mathrm{t}^2}\mathfrak{U}_{\mathfrak{m}_1}(\mathrm{t})d\mathrm{t}]dy.\end{aligned}$$

Similarly, performing the same procedure for the integral to variable y we obtain

$$\begin{aligned}\mathfrak{c}_{\mathfrak{n}\mathfrak{m}} &= \frac{3A\times 2^{-2\mathfrak{k}_1-2\mathfrak{k}_2}}{(1+(\mathfrak{m}_1+1)^2)(1+(\mathfrak{m}_2+1)^2)}\Big[3\int_{-1}^{1}\int_{-1}^{1} \mathrm{t}\mathfrak{W}(\mathrm{t})\mathfrak{U}_{\mathfrak{m}_1}(\mathrm{t})\mathrm{z}\mathfrak{W}(\mathrm{z})\mathfrak{U}_{\mathfrak{m}_2}(\mathrm{z}) \\ &\frac{\partial^2 u(\frac{\mathrm{t}+\tilde{\mathfrak{n}}_1}{2^{\mathfrak{k}_1}},\frac{\mathrm{z}+\tilde{\mathfrak{n}}_2}{2^{\mathfrak{k}_2}})}{\partial \mathrm{t}\partial \mathrm{z}} dzd\mathrm{t} - \frac{1}{2^{\mathfrak{k}_2}}\int_{-1}^{1}\int_{-1}^{1} \mathrm{t}\mathfrak{W}(\mathrm{t})\mathfrak{U}_{\mathfrak{m}_1}(\mathrm{t})\mathfrak{W}^3(\mathrm{z})\mathfrak{U}_{\mathfrak{m}_2}(\mathrm{z})\frac{\partial^3 u(\frac{\mathrm{t}+\tilde{\mathfrak{n}}_1}{2^{\mathfrak{k}_1}},\frac{\mathrm{z}+\tilde{\mathfrak{n}}_2}{2^{\mathfrak{k}_2}})}{\partial \mathrm{t}\partial^2 \mathrm{z}} dzd\mathrm{t}\Big] \\ &- \frac{A\times 2^{-3\mathfrak{k}_1-2\mathfrak{k}_2}}{(1+(\mathfrak{m}_1+1)^2)(1+(\mathfrak{m}_2+1)^2)}\Big[3\int_{-1}^{1}\int_{-1}^{1} \mathfrak{W}^3(\mathrm{t})\mathfrak{U}_{\mathfrak{m}_1}(\mathrm{t})\mathrm{z}\mathfrak{W}(\mathrm{z})\mathfrak{U}_{\mathfrak{m}_2}(\mathrm{z}) \\ &\frac{\partial^3 u(\frac{\mathrm{t}+\tilde{\mathfrak{n}}_1}{2^{\mathfrak{k}_1}},\frac{\mathrm{z}+\tilde{\mathfrak{n}}_2}{2^{\mathfrak{k}_2}})}{\partial^2 \mathrm{t}\partial \mathrm{z}} dzd\mathrm{t} - \frac{1}{2^{\mathfrak{k}_2}}\int_{-1}^{1}\int_{-1}^{1} \mathfrak{W}^3(\mathrm{t})\mathfrak{U}_{\mathfrak{m}_1}(\mathrm{t})\mathfrak{W}^3(\mathrm{z})\mathfrak{U}_{\mathfrak{m}_2}(\mathrm{z})\frac{\partial^4 u(\frac{\mathrm{t}+\tilde{\mathfrak{n}}_1}{2^{\mathfrak{k}_1}},\frac{\mathrm{z}+\tilde{\mathfrak{n}}_2}{2^{\mathfrak{k}_2}})}{\partial^2 \mathrm{t}\partial^2 \mathrm{z}} dzd\mathrm{t}\Big].\end{aligned}$$

Let $u(\mathrm{t},\mathrm{z})$ be a function defined on $[0,1]\times[0,1]$ and $\mathfrak{W}(x)\leq 1$, $max|\frac{\partial^2 u(\frac{\mathrm{t}+\tilde{\mathfrak{n}}_1}{2^{\mathfrak{k}_1}},\frac{\mathrm{z}+\tilde{\mathfrak{n}}_2}{2^{\mathfrak{k}_2}})}{\partial \mathrm{t}\partial \mathrm{z}}|\leq \mathcal{M}_{1,1}$, $max|\frac{\partial^3 u(\frac{\mathrm{t}+\tilde{\mathfrak{n}}_1}{2^{\mathfrak{k}_1}},\frac{\mathrm{z}+\tilde{\mathfrak{n}}_2}{2^{\mathfrak{k}_2}})}{\partial \mathrm{t}\partial^2 \mathrm{z}}|\leq \mathcal{M}_{1,2}$, $max|\frac{\partial^3 u(\frac{\mathrm{t}+\tilde{\mathfrak{n}}_1}{2^{\mathfrak{k}_1}},\frac{\mathrm{z}+\tilde{\mathfrak{n}}_2}{2^{\mathfrak{k}_2}})}{\partial^2 \mathrm{t}\partial \mathrm{z}}|\leq \mathcal{M}_{2,1}$ and $max|\frac{\partial^4 u(\frac{\mathrm{t}+\tilde{\mathfrak{n}}_1}{2^{\mathfrak{k}_1}},\frac{\mathrm{z}+\tilde{\mathfrak{n}}_2}{2^{\mathfrak{k}_2}})}{\partial^2 \mathrm{t}\partial^2 \mathrm{z}}|\leq \mathcal{M}_{2,2}$ where $\mathcal{M}_{1,1},\mathcal{M}_{1,2},\mathcal{M}_{2,1},\mathcal{M}_{2,2}$ are a positive constant and by setting $\mathfrak{W}(\mathrm{z})\leq 1$, we have

$$|\mathfrak{c}_{\mathfrak{n}_1,\mathfrak{m}_1,\mathfrak{n}_2,\mathfrak{m}_2}| \leq \mathfrak{A}[9\mathcal{M}_{1,1}-\frac{3}{2^{\mathfrak{k}_2}}\mathcal{M}_{1,2}-\frac{3}{2^{\mathfrak{k}_1}}\mathcal{M}_{2,1}+\frac{1}{2^{\mathfrak{k}_1+\mathfrak{k}_2}}\mathcal{M}_{2,2}]\int_{-1}^{1}|\mathfrak{U}_{\mathfrak{m}_1}(\mathrm{t})|dt\int_{-1}^{1}|\mathfrak{U}_{\mathfrak{m}_2}(\mathrm{z})|dz,$$

where $\mathfrak{A}=\frac{2^{\frac{-3}{2}(\mathfrak{k}_1+\mathfrak{k}_2)+1}}{\pi(1+(\mathfrak{m}_1+1)^2)(1+(\mathfrak{m}_2+1)^2)}$. Because $\int_{-1}^{1}|U_{\mathfrak{m}}(x)|dx=\int_0^{\pi}|sin(\mathfrak{m}+1)\theta|d\theta\leq\frac{2}{\mathfrak{m}+1}$ so we achieve

$$|\mathfrak{c}_{\mathfrak{n}_1,\mathfrak{m}_1,\mathfrak{n}_2,\mathfrak{m}_2}| \leq \frac{4\mathfrak{A}}{(\mathfrak{m}_1+1)(\mathfrak{m}_2+1)}[9\mathcal{M}_{1,1}-\frac{3}{2^{\mathfrak{k}_2}}\mathcal{M}_{1,2}-\frac{3}{2^{\mathfrak{k}_1}}\mathcal{M}_{2,1}+\frac{1}{2^{\mathfrak{k}_1+\mathfrak{k}_2}}\mathcal{M}_{2,2}].$$

Using $\mathfrak{n}_j\leq 2^{\mathfrak{k}_j-1}\Rightarrow\frac{1}{2^{\mathfrak{k}_j}}\leq\frac{1}{2\mathfrak{n}_j}, j=1,2$, we get

$$\begin{aligned}|\mathfrak{c}_{\mathfrak{n}_1,\mathfrak{m}_1,\mathfrak{n}_2,\mathfrak{m}_2}| &\leq \frac{1}{\mathfrak{m}_1{}^3\mathfrak{m}_2{}^3}[\frac{9\mathcal{M}_{1,1}}{(\mathfrak{n}_1\mathfrak{n}_2)^{\frac{3}{2}}}-\frac{3\mathcal{M}_{1,2}}{2(\mathfrak{n}_1)^{\frac{3}{2}}(\mathfrak{n}_2)^{\frac{5}{2}}}-\frac{3\mathcal{M}_{2,1}}{2(\mathfrak{n}_1)^{\frac{5}{2}}(\mathfrak{n}_2)^{\frac{3}{2}}}+\frac{\mathcal{M}_{2,2}}{(\mathfrak{n}_1\mathfrak{n}_1)^{\frac{5}{2}}}] \\ &\leq \frac{[9\mathcal{M}_{1,1}+\mathcal{M}_{2,2}]}{\mathfrak{m}_1{}^3\mathfrak{m}_2{}^3(\mathfrak{n}_1\mathfrak{n}_2)^{\frac{3}{2}}(\mathfrak{n}_1\mathfrak{n}_2)^{\frac{3}{2}}}\leq\frac{\widetilde{\mathcal{M}}}{\mathfrak{m}_1{}^3\mathfrak{m}_2{}^3(\mathfrak{n}_1\mathfrak{n}_2)^{\frac{3}{2}}(\mathfrak{n}_1\mathfrak{n}_2)^{\frac{3}{2}}}.\end{aligned}$$

This means that the series (8) is absolutely convergence and the theorem 1 results in the expansion of the $u(x,y)$ converges uniformly.

6. Numerical examples

We demonstrate the proposed method in the previous section to show the efficiency and to obtain the numerical solution nonhomogeneous or homogeneous partial differential equations. We solve two examples from [12] and two examples from [19] and all the calculations are accomplished using the Mathematica software. To compare exact and numerical solutions of the reported method, we choose the maximum error $L_\infty=max|u(x_p,y_q)-\tilde{u}(x_p,y_q)|$ and relative error $\frac{\|u(x_p,y_q)-\tilde{u}(x_p,y_q)\|_2}{\|u(x_p,y_q)\|_2}, p=1,2,\ldots,\mathfrak{N}_1, q=1,2,\ldots,\mathfrak{N}_2$.

Example 1 *The fractional Poisson equation (2) approximated with the homogeneous boundary conditions and* $f(x,y)=3x^{\frac{2}{3}}y(y-1)/\Gamma(\frac{2}{3})+4\sqrt{y}x(x-1)/\sqrt{\pi}$. *The exact solution of this problem is* $u(x,y)=xy(x-1)(y-1),(x,y)\in[0,1]\times[0,1]$. *Table 1 indicates the relative error and maximum absolute error (MAE) or* L_∞ *of all collocation points by applying different values of* $\mathfrak{m}_1=\mathfrak{m}_2=5$, *scale degree of* $\mathfrak{k}_1=\mathfrak{k}_2=1$ *and the values of fractional order* $\alpha=\frac{4}{3},\beta=\frac{3}{2}$. *Also, the current method compares with absolute error in paper [12].*

Example 2 *The fractional Poisson equation (2) approximated with the homogeneous boundary conditions and* $f(x,y)=8x^{\frac{3}{2}}y^3(-5y+5+8xy-8x)/5\sqrt{\pi}+27y^{\frac{5}{3}}x^3(-2x+2+3xy-3y)/10\Gamma(\frac{2}{3})$. *The exact solution of this problem is* $u(x,y)=x^3y^3(1-x)(1-y),(x,y)\in[0,1]\times[0,1]$. *Table 1 indicates the relative error and maximum absolute error (MAE) or* L_∞ *of all collocation points by applying different values of* $\mathfrak{m}_1=\mathfrak{m}_2=5$, *scale degree of* $\mathfrak{k}_1=\mathfrak{k}_2=1$ *and the values of fractional order* $\alpha=\frac{3}{2},\beta=\frac{4}{3}$. *Also, the current method compares* *with absolute error in paper [12]. The graphs of numerical method are shown in Figure 0 for* $ɱ_1 = ɱ_2 = 5, ꝁ_1 = ꝁ_2 = 2$ *(left side) and* $ɱ_1 = ɱ_2 = 5, ꝁ_1 = 2, ꝁ_2 = 1$ *(right side).*

Example 3 *The fractional partial differential equation (1) approximated with the initial conditions* $u(0,t) = u(x,0) = 0$ *and* $g(x,t) = \frac{4x^{\frac{3}{4}}t + xt^{\frac{3}{4}}}{3\Gamma(\frac{3}{4})}$. *The exact solution of this problem is* $u(x,t) = xt, (x,y) \in [0,1] \times [0,1]$. *Table 2 indicates the relative error and maximum absolute error (MAE) or* L_∞ *of all collocation points by applying different values of* $ɱ_1, ɱ_2$, *scale degree of* $ꝁ_1, ꝁ_2$ *and the values of fractional order* $\alpha = \beta = \frac{1}{4}$. *The abbreviation DP and TP of Table 2 and Table 3 are dilation and translation parameter respectively. Also, the current method compares with absolute error in paper [19].*

Example 4 *The fractional partial differential equation (1) approximated with the initial conditions* $u(0,t) = u(x,0) = 0$ *and* $g(x,t) = \frac{9x^2t^{\frac{5}{3}}}{5\Gamma(\frac{2}{3})} + \frac{8x^{\frac{3}{2}}t^2}{3\Gamma(\frac{1}{2})}$. *The exact solution of this problem is* $u(x,t) = x^2t^2, (x,y) \in [0,2) \times [0,2)$. *Table 3 indicates the relative error and maximum absolute error (MAE) or* L_∞ *of all collocation points by applying different values of* $ɱ_1, ɱ_2$, *scale degree of* $ꝁ_1, ꝁ_2$ *and the values of fractional order* $\alpha = \frac{1}{3}, \beta = \frac{1}{2}$. *Also, Table 4 indicates the numerical approximate and maximum absolute error (MAE) of all collocation points by applying different values scale degree and* $ɲ_1 = ɲ_2 = 16$. *The current method compares with absolute error with paper [19] in Table 4. The graphs of numerical method are shown in Figure 1 for* $ɱ_1 = ɱ_2 = 2, ꝁ_1 = 2, ꝁ_2 = 1$ *(left side) and* $ꝁ_1 = ꝁ_2 = 2$ *(right side).*

Table 1: Approximate errors for some different points for Examples 1 and 2, for $ɱ_1 = ɱ_2 = 5, ꝁ_1 = ꝁ_2 = 1$.

22cm(x_p, y_q)	*Ex1*		*Ex2*	
	$L_\infty forEx1$	$L_\infty inheydari$	$L_\infty forEx2$	$L_\infty inheydari$
(0.1,0.1)	$1.02E-17$	$2.51E-16$	$7.08E-18$	$6.54E-17$
(0.3,0.3)	$6.94E-18$	$5.52E-17$	$2.05E-17$	$6.68E-17$
(0.5,0.5)	$1.39E-17$	$7.73E-17$	$6.26E-18$	$3.54E-17$
(0.7,0.7)	$3.47E-18$	$5.53E-17$	$1.56E-17$	$3.50E-17$
(0.9,0.9)	$4.92E-18$	$1.44E-17$	$3.98E-18$	$2.50E-17$

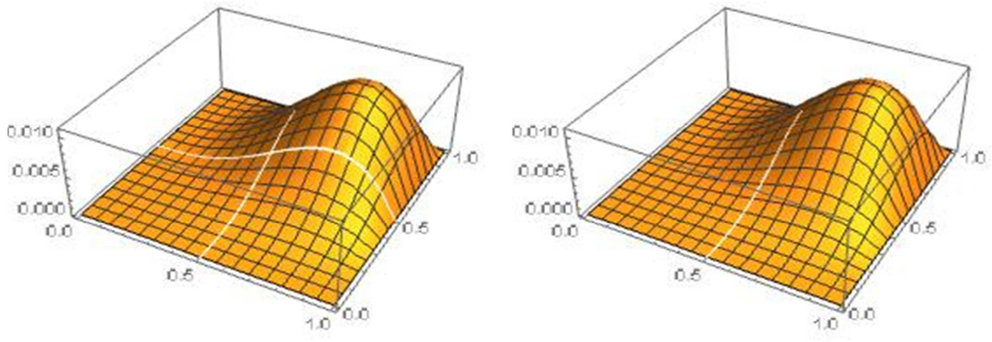

Figure 1: Numerical Solution for Example 4 with $ɲ_1 = ɲ_2 = 10$ (left) and $ɲ_1 = 10, ɲ_2 = 5$ (right).

Table 2: The numerical errors of different $\mathfrak{m}_1,\mathfrak{m}_2,\mathfrak{t}_1 and \mathfrak{t}_2$ for Example 3.

DP	TP	$\mathfrak{n}_1,\mathfrak{n}_2$	h_p,h_q	$Relative error$	MAE
1,1	4,4	4,4	1/4,1/4	$8.47852E-15$	$4.57967E-15$
	6,6	6,6	1/6,1/6	$8.81142E-15$	$6.43929E-15$
	8,8	8,8	1/8,1/8	$7.02004E-15$	$5.88418E-15$
2,2	2,2	4,4	1/4,1/4	$1.72491E-13$	$1.55875E-13$
	3,3	6,6	1/6,1/6	$1.89458E-13$	$1.26898E-13$
	4,4	8,8	1/8,1/8	$5.32139E-12$	$3.54788E-12$
2,1	6,3	12,3	1/12,1/3	$7.04604E-13$	$3.85445E-13$
3,1	4,4	16,4	1/16,1/4	$1.17783E-13$	$1.04249E-13$
4,1	4,4	32,4	1/32,1/4	$1.46955E-12$	$1.46833E-12$

Table 3: The numerical errors of different $\mathfrak{m}_1,\mathfrak{m}_2,\mathfrak{t}_1 and \mathfrak{t}_2$ for Example 4.

DP	TP	$\mathfrak{n}_1,\mathfrak{n}_2$	h_p,h_q	$Relative error$	MAE
1,1	3,3	3,3	1/3,1/3	$3.29631E-2$	$1.23018E-2$
	4,4	4,4	1/4,1/4	$3.5306E-2$	$1.46436E-2$
	5,5	5,5	1/5,1/5	$3.53892E-2$	$1.63514E-2$
	6,6	6,6	1/6,1/6	$3.419024E-2$	$1.79463E-2$
	7,7	7,7	1/7,1/7	$3.46035E-2$	$1.84352E-2$
	8,8	8,8	1/8,1/8	$3.50882E-2$	$1.94350E-2$
2,2	2,2	4,4	1/4,1/4	$3.61879E-2$	$1.63473E-2$
	3,3	6,6	1/6,1/6	$3.74160E-2$	$1.92758E-2$
	4,4	8,8	1/8,1/8	$3.66395E-2$	$1.9296E-2$
	5,5	10,10	1/10,1/10	$3.71234E-2$	$2.04518E-2$

Table 4: The numerical error of different points for $\mathfrak{m}_i,\mathfrak{t}_i, i=1,2$ for Example 4.

31cm(x_p,t_q)		*Maximum Absolute Error*		
	Exact	$\mathfrak{m}_1=\mathfrak{m}_2=8$	$\mathfrak{m}_1=\mathfrak{m}_2=4$	$\mathfrak{m}_1=\mathfrak{m}_2=2$
	solution	$\mathfrak{t}_1=\mathfrak{t}_2=1$	$\mathfrak{t}_1=\mathfrak{t}_2=2$	$\mathfrak{t}_1=\mathfrak{t}_2=3$
$(\frac{1}{16},\frac{1}{16})$	$1.52588E-5$	$3.76035E-7$	$4.75427E-7$	$7.94960E-7$
$(\frac{3}{16},\frac{3}{16})$	$1.23596E-3$	$1.59657E-5$	$1.55934E-5$	$1.13359E-5$
$(\frac{5}{16},\frac{5}{16})$	$9.53674E-3$	$1.00622E-4$	$1.01199E-4$	$1.83140E-4$
$(\frac{7}{16},\frac{7}{16})$	$3.66364E-2$	$3.33133E-4$	$3.31909E-4$	$4.39544E-4$
$(\frac{9}{16},\frac{9}{16})$	$1.00113E-1$	$8.07738E-4$	$8.48683E-4$	$1.39579E-3$
$(\frac{11}{16},\frac{11}{16})$	$2.23404E-1$	$1.62873E-3$	$1.69392E-3$	$2.33208E-3$
$(\frac{13}{16},\frac{13}{16})$	$4.35806E-1$	$2.90607E-3$	$3.01301E-3$	$4.77224E-3$
$(\frac{15}{16},\frac{15}{16})$	$7.24476E-1$	$4.76018E-3$	$4.91540E-3$	$6.86708E-3$

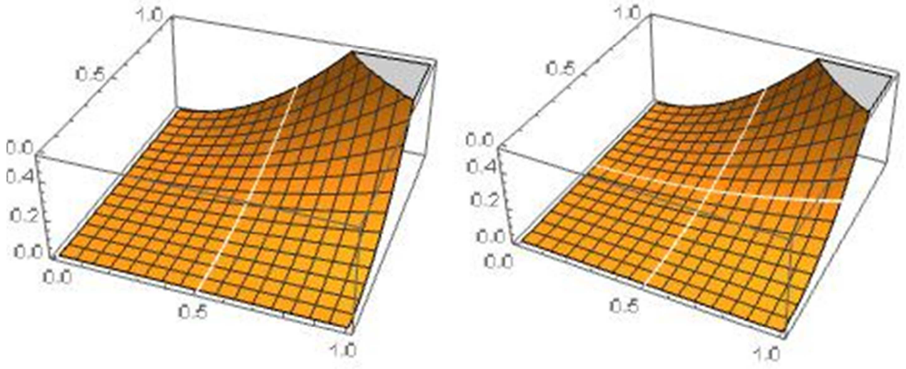

Figure 2: Numerical Solution for Example 4 with $\mathfrak{m}_1 = \mathfrak{m}_2 = 2, \mathfrak{k}_1 = 2, \mathfrak{k}_2 = 1$ (left) and $\mathfrak{k}_1 = \mathfrak{k}_2 = 2$ (right).

7. Conclusion

In this article, the collocation method of the two-dimensional Chebyshev wavelet of the second kind is used to solving the fractional partial differential equation on $[0,1] \times [0,1]$. The properties of the Chebyshev wavelet basis are orthonormal and small intervals of support. They can establish a sparse coefficients matrix that these properties simplify computing. Moreover, we investigated the convergence analysis of the collocation method. It is proved that the approximation solution is convergence uniformly. The collocation method was shown very convenient for solving boundary and initial value problems.

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